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Superconductivity in the Hubbard model

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Abstract. A non-conventional theory for superconductivity that is not based on a Fermi liquid description is presented. Using a functional integral method we show that the two-dimensional Hubbard model coupled weakly in the third dimension has a superconducting solution for the non-half-filled band case. The superconducting critical temperature is determined by the Bose–Einstein condensation temperature, which increases with the coupling to the third dimension and with the hole concentration. The critical behaviour is similar to that of a three-dimensional type II superconductor with a neutral Higgs field induced by spinon fluctuations. In the language of the lattice gauge theory we find that the spontaneous breaking of the U(1) local gauge symmetry in the quantum paramagnet confining phase leads to superconductivity.

1. Introduction

Experimental results indicate that correlation effects play an important role in the new superconductors [1]. These correlations are described by a Hubbard model. This model uses two parameters: U, the effective on-site interelectronic Coulomb repulsion, and $8t$, the band width (band structure calculations show that the band arises from the anti-bonding combination of the Cu $d_{x^2-y^2}$ and O $p\sigma$ orbitals). The Hubbard model has been mapped onto a two-dimensional (2D) Heisenberg antiferromagnet (AF) [2]; this mapping is exact for large U, and for small values of U it is justified by scaling. Therefore, exactly at the half-filled band ($n_\uparrow + n_\downarrow = 1$, $\delta = 0$), we have a 2D AF. By doping the system ($n_\uparrow + n_\downarrow < 1$, $\delta \neq 0$), the presence of holes transform the AF insulator to a new type of liquid [3–10]† that can give rise to superconductivity [3–9].

In spite of these successes recent Monte Carlo calculations performed by Hirsch [10] have indicated difficulties with superconductivity in the 2D Hubbard model, so it is important to investigate the different approximations used.

To my knowledge there are two approximations: the first one neglects phase fluctuations [3–7] of the resonant valence bond (RVB) order parameter, and the second approximates the effective hopping term by $t\delta$, where δ is the hole concentration. This approximation is crude, since after the elimination of the double occupied sites the hopping term becomes a two-body interaction term. The slow variation in the RVB order parameter was considered at mean-field levels and it was shown that we can have either d- or s-wave superconductivity [7].

In order to study these problems we compute the free energy in terms of amplitudes and phases. For the amplitude part, we find that the effective hopping term behaves as

† In [5] the author uses two sublattices. In [8] the authors work in the small-U/t limit.

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a bosonic hole–hole correlation function. In a mean-field approximation this correlation is non-zero in the case we have a Bose condensation. Since there is no condensation in 2D we argue that a weak coupling in the third dimension is required. Combining the requirements of boson condensation with the non-vanishing of the RVB order parameter, we find that the mean-field critical superconducting temperatures increase with hole concentration and with coupling in the third dimension. The dependence on the third dimension is model-dependent [25]; we can have hopping in the third dimension or Josephson-like coupling between copper oxide planes. Here we limit ourselves to the first possibility.

The phase-dependent part is similar to the 3D x–y model coupled to a gauge field [11]. As a result of this gauge field we have two correlation lengths, the usual superconducting correlation length ξ_s, and the Higgs correlation length ξ_H [11] induced by the spinon fluctuations.

This 3D x–y model coupled to a gauge field has some similarities to the type II superconductor. ξ_H represents the penetration length, and decreases with increases in hole concentration when $\xi_H \to \infty$ superconductivity is destroyed. Specifically, we identify ξ_H with the hole boson correlation length and ξ_s with the RVB correlation length.

The format of this paper is as follows. In § 2 we discuss the Hubbard model and show how it can be described in terms of new bosons and fermions. Section 3 is devoted to bosonisation in terms of the RVB order parameter and density order parameter. In § 4 we compute the effective potential in terms of amplitude and phase, and in § 5 we estimate the superconducting critical temperature obtained by demanding that T_c^{RVB} (the RVB critical temperature) is equal to T_c^{BE} (the Bose–Einstein condensation temperature). Finally, in § 6 we discuss the phase-dependent part, based on existing results in lattice gauge theory.

2. The model

We consider the Hubbard model of a cubic square lattice

$$H = -t \sum_{\sigma=\uparrow,\downarrow} \sum_{n,\mu} C_\sigma^+(n)C_\sigma(n+\mu) + \text{HC} + U\sum_n C_\uparrow^+(n)C_\uparrow(n)C_\downarrow^+(n)C_\downarrow(n)$$
$$- \varepsilon_F \sum_n [C_\uparrow^+(n)C_\uparrow(n) + C_\downarrow^+(n)C_\downarrow(n)]. \qquad (1a)$$

The investigation of this model will be performed with the aid of the slave boson method [4, 6, 7, 12–16]. Formally our derivation is similar to that given in [4], but conceptually we follow Schwinger's ideas [17]. The basic thing is to replace a nonlinear operator by a given combination of bosonic or fermionic harmonic oscillators. A known example is the angular momentum operator which is replaced by two harmonic oscillators (HOs). The Hilbert space of the angular momentum is replaced by a direct product of the two HOs. For the electron operators we use two bosonic HOs and two fermionic HOs. We introduce for each electron operator two bosons e, e^+ (for holes), d, d^+ (for double occupancy); and two fermions $f_\uparrow, f_\uparrow^+$ (spin up) and $f_\downarrow, f_\downarrow^+$ (spin down). As a result the Hilbert space of the electron which is spanned by four states $|0\rangle$, $|\uparrow\rangle$, $|\downarrow\rangle$, $|\uparrow\downarrow\rangle$, is replaced by a direct product given by $|e, d, f_\uparrow, f_\downarrow)$. The space of each variable is spanned by the following vectors: $|e) = \{|0e), |1e)\}$; $|d) = \{|0d), |1d)\}$; $|f_\uparrow) = \{|0_\uparrow), |1_\uparrow)\}$, and $|f_\downarrow) = \{|0_\downarrow), |1_\downarrow)\}$ [$|0e), |0d), |0_\uparrow), |0_\downarrow)$ is the ground state and $|1e), |1d), |1_\uparrow), |1_\downarrow)$ is the first excited state]. Each original state is replaced by a given combination of four HO states: $|0\rangle \to |1e, 0d, 0_\uparrow, 0_\downarrow)$, $|\uparrow\rangle \to |0e, 0d, 1_\uparrow, 0_\downarrow)$, $|\downarrow\rangle \to |0e, 0d, 0_\uparrow, 1_\downarrow)$, and

$|\downarrow\uparrow\rangle \to |0e, 1d, 0_\uparrow, 0_\downarrow)$.

On this basis we establish the following relations:

$$C_\sigma^+(n) = e(n)f_\sigma^+(n) + \sigma d^+(n)f_{-\sigma}(n) \qquad C_\sigma(n) = [C_\sigma^+(n)]^+ \qquad \sigma = \uparrow, \downarrow. \tag{1b}$$

The e, e^+, d, d^+ satisfies the commutation relation and the f_σ, f_σ^+ anticommutation. Using these definitions we obtain that the C_σ, C_σ^+ satisfies anticommutation relation.

The completeness of the original states ($|0\rangle\langle 0| + |\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow| + |\uparrow\downarrow\rangle\langle\uparrow\downarrow| = 1$) is replaced by

$$e^+(n)e(n) + d^+(n)d(n) + \sum_{\sigma=\uparrow,\downarrow} f_\sigma^+(n)f_\sigma(n) = 1. \tag{1c}$$

It is important to note that the number of variables on the right-hand side of equation (1b) is larger than on the left. We transform four operators into eight. In order to have a correct transformation we need constraint equations. Equation (1c) provides the first one, and in addition we have $(e^+)^2 = (d^+)^2 = ed^+ = 0$.

Using these relations and equations (1a–c) we obtain the following form for the Hubbard Hamiltonian:

$$\begin{aligned} H = -t\sum_{n,\mu}\sum_{\sigma=\uparrow,\downarrow} &\{[e(n)e^+(n+\mu) - d^+(n)d(n+\mu)]f_\sigma^+(n)f_\sigma(n+\mu) \\ &+ [e(n)d(n+\mu) + e(n+\mu)d(n)]\sigma f_\sigma^+(n)f_{-\sigma}^+(n+\mu) + \mathrm{HC}\} \\ &+ (U-\varepsilon_\mathrm{F})\sum_n d^+(n)d(n) + \varepsilon_\mathrm{F}\sum_n e^+(n)e(n) - \varepsilon_\mathrm{F}\sum_n 1. \end{aligned} \tag{2a}$$

Here n runs over the square lattice points and μ is the unity vector [$n = (n_x, n_y)$, $\mu = \{(0, \pm a_y), (\pm a_x, 0)\}$].

Using the coherent state representation we construct a path integral in terms of the boson and fermion variables [18]. In this representation $f_\uparrow, f_\uparrow^+, f_\downarrow, f_\downarrow^+$ satisfies the Grassman algebra and e, e^+, d, d^+ become complex numbers which, in addition to normal bosons, satisfy $(e)^2 = (e^+)^2 = (d)^2 = (d^+)^2 = ed^+ = e^+d = 0$. Contrary to normal coherent states the hole boson coherent state $|Z)$ is given by $|Z) = \exp(Ze^+)|0e) = |0e) + Z|1e)$, since $Z^2 = 0$.

The path integral for the Hamiltonian given in equation (2a) takes the form [19]

$$Z = \int \mathrm{D}f_\uparrow^+\, \mathrm{D}f_\uparrow\, \mathrm{D}f_\downarrow^+\, \mathrm{D}f_\downarrow\, \mathrm{D}e^+\, \mathrm{D}e\, \mathrm{D}d^+\, \mathrm{D}d\, \mathrm{D}\lambda\, \exp(A). \tag{2b}$$

The integration with respect to the field λ was introduced in order to satisfy equation (1c). The action A which appears in the path integral is given by

$$\begin{aligned} A = \int_0^\beta \mathrm{d}\tau \Bigg(&\sum_{n,\sigma=\uparrow,\downarrow} f_\sigma^+(n;\tau)[\partial_\tau - \lambda(n;\tau)]f_\sigma(n;\tau) \\ &+ t\sum_{n,\mu,\sigma=\uparrow,\downarrow} \{[e(n;\tau)e^+(n+\mu;\tau) - d^+(n;\tau)d(n+\mu;\tau)] \\ &\times f_\sigma^+(n;\tau)f_\sigma(n+\mu;\tau) + [e(n;\tau)d(n+\mu;\tau) + e(n+\mu;\tau)d(n;\tau)] \\ &\times \sigma f_\sigma^+(n;\tau)f_{-\sigma}^+(n+\mu;\tau) + \mathrm{HC}\} + \sum_n d^+(n;\tau) \\ &\times [\partial_\tau + U - \varepsilon_\mathrm{F} - \lambda(n;\tau)]d(n;\tau) + \sum_n e^+(n;\tau) \\ &\times [\partial_\tau - \varepsilon_\mathrm{F} - \lambda(n;\tau)]e(n;\tau) + \sum_n [\varepsilon_\mathrm{F} + \lambda(n;\tau)]\Bigg). \end{aligned} \tag{2c}$$

In the limit $U \gg t$ one integrates out the d, d^+ variables and equation (2c) reduces to [2, 4, 6]:

$$A' = \int_0^\beta \mathrm{d}\tau \Big[\sum_{n,\sigma=\uparrow,\downarrow} f_\sigma^+(n';\tau)[\partial_\tau - \lambda(n;\tau)] f_\sigma(n;\tau)$$
$$+ J \sum_{n,\mu} \Big(\sum_{\sigma=\uparrow,\downarrow} \sigma f_\sigma^+(n;\tau) f_{-\sigma}^+(n+\mu;\tau) \Big) \Big(\sum_{\sigma=\uparrow,\downarrow} \sigma f_\sigma^+(n;\tau) f_{-\sigma}^+(n+\mu;\tau) \Big)^+$$
$$+ t \sum_{n,\mu,\sigma=\uparrow,\downarrow} [f_\sigma^+(n;\tau) e(n;\tau) e^+(n+\mu;\tau) f_\sigma(n+\mu;\tau) + \mathrm{HC}]$$
$$+ \sum_n e^+(n;\tau)[\partial_\tau - \varepsilon_\mathrm{F} - \lambda(n;\tau)] e(n;\tau) \Big] \tag{2d}$$

where $J = 4t^2/U$ and β is the inverse temperature ($\beta = 1/T, K_\mathrm{B} = 1, \hbar = 1$), f_σ, f_σ are the spinon and e, e^+ the holon bosons [5].

3. The bosonisation of the action

The bosonisation method is based on replacing pairs of fermions with collective variables that might not be real bosons (see the case of pairons introduced by Schrieffer [20] for the BCS order parameter). The identification of the collective variables is done within a mean-field approximation.

Considering the actions (2c) and (2d) we can use the following approximations:

$$\sum_{\sigma=\uparrow,\downarrow} e(n)d(n+\mu)\sigma f_\sigma^+(n) f_{-\sigma}^+(n+\mu) \simeq \sum_{\sigma=\uparrow,\downarrow} [\langle e(n)d(n+\mu)\rangle \sigma f_\sigma^+(n) f_{-\sigma}^+(n+\mu)$$
$$+ e(n)d(n+\mu)\langle \sigma f_\sigma^+(n) f_{-\sigma}^+(n+\mu)\rangle - \langle e(n)d(n+\mu)\rangle\langle \sigma f_\sigma^+(n) f_{-\sigma}^+(n+\mu)\rangle]$$

in equation (2c) and

$$\Big(\sum_{\sigma=\uparrow,\downarrow} \sigma f_\sigma^+(n) f_{-\sigma}^+(n+\mu) \Big) \Big(\sum_{\sigma=\uparrow,\downarrow} \sigma f_\sigma^+(n) f_{-\sigma}^+(n+\mu) \Big)^+$$
$$\simeq \Big\langle \Big(\sum_\sigma \sigma f_\sigma^+(n) f_{-\sigma}^+(n+\mu) \Big) \Big\rangle \Big(\sum_{\sigma=\uparrow,\downarrow} \sigma f_\sigma^+(n) f_{-\sigma}^+(n+\mu) \Big)^+ + \mathrm{HC}$$
$$- \Big\langle \Big(\sum_{\sigma=\uparrow,\downarrow} \sigma f_\sigma^+(n) f_{-\sigma}^+(n+\mu) \Big) \Big\rangle \Big\langle \Big(\sum_\sigma \sigma f_\sigma^+(n) f_{-\sigma}^+(n+\mu) \Big)^+ \Big\rangle$$

in equation (2d).

It is important to note (see the first equation) that $\langle e(n)d(n+\mu)\rangle \neq 0$, in spite of the fact that $\langle d \rangle = 0$. The reason is that the average is performed with respect to a Bose Hamiltonian which contains a term $e(n)d(n+\mu)\langle f_\sigma^+(n) f_{-\sigma}^+(n+\mu)\rangle$. A similar decoupling is also performed for the density part:

$$e(n)e^+(n+\mu) f_\sigma^+(n) f_\sigma(n+\mu)$$
$$\simeq \langle e(n)e^+(n+\mu)\rangle f_\sigma^+(n) f_\sigma(n+\mu) + e(n)e^+(n+\mu)\langle f_\sigma^+(n) f_\sigma(n+\mu)\rangle$$
$$- \langle e(n)e^+(n+\mu)\rangle\langle f_\sigma^+(n) f_\sigma(n+\mu)\rangle.$$

As a result of these approximations we diagonalise a Bose and Fermi Hamiltonian in the presence of a background field.

In order to go beyond the mean field we will use the method of functional integral for collective variables [19]. At the level of a stationary phase approximation this method reduces to the mean-field result. In the action given by equation (2c) we have two collective variables (and respectively their complex conjugates)

$$B^*(n, n+\mu) = \sum_{\sigma=\uparrow,\downarrow} \sigma f_\sigma^+(n) f_{-\sigma}^+(n+\mu) \tag{2e}$$

$$T_\mu^*(n) = \sum_{\sigma=\uparrow,\downarrow} f_\sigma^+(n) f_\sigma(n+\mu). \tag{2f}$$

The first one is the RVB order parameter [1, 2] and the second represents the spinon gauge field. The substitution of the collective variables is performed with the new constraint fields $\rho_\mu(n)\rho^*_\mu(n)$ for the substitution of equation (2*f*) and $q(n, n+\mu)$ $[q^*(n, n+\mu)]$ for the substitution of equation (2*e*). The method for doing this is based on the following identity:

$$\exp(AB) = \int_{-\infty}^{\infty} \mathrm{d}x \exp(Ax)\delta(x-B) = \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}y \exp(Ax)\exp[-\mathrm{i}y(x-B)]. \qquad (2g)$$

The situation for the action given in equation (2*d*) is simpler since the collective variable

$$\sum_{\sigma=\uparrow,\downarrow} \sigma f^+_\sigma(n) f^+_{-\sigma}(n+\mu)$$

is introduced by the Hubbard–Stratonovici [19] method. The interaction

$$J\Bigg(\sum_{\sigma=\uparrow,\downarrow} \sigma f^+_\sigma(n) f^+_{-\sigma}(n+\mu)\Bigg)\Bigg(\sum_{\sigma=\uparrow,\downarrow} \sigma f^+_\sigma(n) f^+_{-\sigma}(n+\mu)\Bigg)^+$$

is replaced by a Gaussian integral in the action over the field $q(n, n+\mu)$:

$$\Bigg(\sum_{\sigma=\uparrow,\downarrow} \sigma f^+_\sigma(n) f^+_{-\sigma}(n+\mu)\Bigg) q(n,n+\mu) + \Bigg(\sum_{\sigma=\uparrow,\downarrow} \sigma f^+_\sigma(n) f^+_{-\sigma}(n+\mu)\Bigg)^+ q^*(n,n+\mu)$$
$$- \frac{1}{2J} q(n,n+\mu) q^*(n,n+\mu).$$

A convenient way to write the actions (2*c*) and (2*d*) in terms of the collective field is to use Dirac matrices:

$$\boldsymbol{\gamma} = \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix} \qquad \boldsymbol{\Sigma} = \begin{pmatrix} \sigma, & 0 \\ 0, & \sigma \end{pmatrix} \qquad \boldsymbol{\alpha} = \begin{pmatrix} \sigma & 0 \\ 0 & -\sigma \end{pmatrix}$$
$$\boldsymbol{\gamma}^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \qquad \hat{\mathbf{I}} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \qquad (3a)$$

where $\mathbf{I}$, $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the 2×2 Pauli matrices and $\boldsymbol{\gamma}$, $\boldsymbol{\Sigma}$, $\boldsymbol{\alpha}$ are 4×4 matrices constructed from the Pauli matrices $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$, $\boldsymbol{\Sigma} = (\Sigma_1, \Sigma_2, \Sigma_3)$, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$.

In the addition we introduce the spinors

$$\psi^+ = (f^+_\uparrow, f^+_\downarrow, f_\uparrow, f_\downarrow) \qquad \psi = (\psi^+)^+$$

and

$$\varphi^+ = (e^+, d^+, e, d) \qquad \varphi = (\varphi^+)^+.$$

For equation (2*d*) we have only one boson field e, e^+ and we do not use the spinor ϕ, ϕ^+.

We define new matrices as a function of the original ones given in equation (3*a*)

$$\boldsymbol{\gamma}^\pm_0 = \tfrac{1}{2}(\gamma_0 \pm \hat{I}) \qquad \boldsymbol{\gamma}^\pm = \tfrac{1}{2}(\alpha_2 \pm \gamma_2)$$
$$\boldsymbol{\Sigma}^\pm_3 = (\Sigma_3 \pm \gamma_0) \qquad \boldsymbol{\alpha}^\pm = \tfrac{1}{2}(\alpha_1 \pm \gamma_1). \qquad (3b)$$

An additional convention will be to include in μ the vector $\mu = (0, 0)$ $[\mu = \{(0,0), (\pm a_x, 0), (0, \pm a_y)\}]$. Using these definitions we replace the action A given in equation (2*c*) by

$$A = A_\mathrm{F} + A_\mathrm{B} + A_\mathrm{C} \qquad (4a)$$

where A_F is the fermion part, A_B is the boson part and A_C is the collective part.

$$A_\mathrm{F} = \int_0^\beta \mathrm{d}\tau \sum_{n,\mu} \psi^+(n;\tau)\{\delta_{n,n+\mu}\, \hat{I}[\partial_\tau - \lambda(n,\tau)] + (1 - \delta_{n,n+\mu})$$

$$\times [\gamma_0^+ \rho_\mu(n;\tau) + \gamma_0^- \rho_\mu^*(n,\tau) + \gamma^+ q(n, n+\mu;\tau) + \gamma^- q^*(n, n+\mu;\tau)]\}\psi(n+\mu;\tau) \tag{4b}$$

and

$$A_{\mathrm{B}} = \int_0^\beta \mathrm{d}\tau \sum_{n,\mu} \varphi^+(n;\tau)[\![\delta_{n,n+\mu}\{\gamma_0\partial_\tau - \hat{I}[\tfrac{1}{2}U + \lambda(n;\tau)] - \Sigma_3(\varepsilon_{\mathrm{F}} - \tfrac{1}{2}U)\} + t(1-\delta_{n,n+\mu})[\Sigma_3^+ T_\mu(n,\tau) + \Sigma_3^- T_\mu^*(n,\tau) + \alpha^+ B(n,n+\mu;\tau) + \alpha^- B^*(n,n+\mu;\tau)]]\!]\varphi(n+\mu;\tau) \tag{4c}$$

with the collective part

$$A_{\mathrm{C}} = \int_0^\beta \mathrm{d}\tau \sum_{n,\mu} \{\delta_{n,n+\mu}[\varepsilon_{\mathrm{F}} + \lambda(n;\tau)] - (1-\delta_{n,n+\mu}) + [\rho_\mu(n;\tau)T_\mu^*(n;\tau) + q(n,n+\mu;\tau)B^*(n,n+\mu;\tau) + \mathrm{CC}]\}. \tag{4d}$$

The action given in equation (2*d*) can be expressed in a similar form:

$$A = A_{\mathrm{F}}' + A_{\mathrm{B}}' + A_{\mathrm{C}}' \tag{4e}$$

where

$$A_{\mathrm{F}}' = A_{\mathrm{F}} \tag{4f}$$

$$A_{\mathrm{B}}' = \int_0^\beta \mathrm{d}\tau \left[\sum_n e^+(n;\tau)[\partial_\tau - \varepsilon_{\mathrm{F}} - \lambda(n;\tau)]e(n;\tau) + \left(t\sum_{n,\mu}(1-\delta_{n,n+\mu})e(n,\tau)T_\mu(n;\tau)e^+(n+\mu;\tau) + \mathrm{HC}\right)\right] \tag{4g}$$

$$A_{\mathrm{C}}' = \int_0^\beta \mathrm{d}\tau \sum_{n,\mu}\{\delta_{n,n+\mu}[\varepsilon_{\mathrm{F}} + \lambda(n;\tau)] - (1-\delta_{n,n+\mu})[(1/2J)q(n,n+\mu;\tau)q^*(n,n+\mu;\tau) + \rho_\mu(n;\tau)T_\mu^*(n;\tau) + \rho_\mu^*(n;\tau)T_\mu(n;\tau)]\}. \tag{4h}$$

The superconducting order parameter is represented in terms of the collective variables $\langle\Sigma\sigma f_\sigma^+(n)f_\sigma^+(n+\mu)\rangle$ and the hole boson variables $e(n)$

$$\left\langle\sum_\sigma \sigma C_\sigma^+(n)C_{-\sigma}^+(n+\mu)\right\rangle \simeq \langle e(n)e(n+\mu)\rangle$$

$$\left\langle\sum_\sigma \sigma f_\sigma^+(n)f_{-\sigma}^+(n+\mu)\right\rangle \simeq \langle e(n)\rangle\langle e(n+\mu)\rangle\langle q^*(n,n+\mu)\rangle. \tag{5a}$$

This last expression shows that the SC order parameter is nonzero when both $\langle q\rangle \neq 0$ and $\langle e\rangle \neq 0$. This means that in addition to the non-vanishing of the RVB order parameter $\langle q\rangle \neq 0$, we must have condensation of holes, $\langle e\rangle \neq 0$. A different possibility that would exclude boson condensation but would allow pairing of hole bosons such that $\langle e(n)e(n+\mu)\rangle \neq 0$, $\langle e\rangle = 0$ would be possible if we had a direct attraction of bosons [21].

In order to take care of the boson condensation we shift the hole boson operator by a macroscopic classical field and look for the minimum of the action given in equations (4*a*) and (4*e*) with respect to the variation of the classical fields v, v^*. Formally, we shift the hole variables $e \to e + v$, $e^+ \to e^+ + v^*$, as a result of which the action given in equations (4*a*) and (4*e*) changes to:

$$A \to A + \int_0^\beta \mathrm{d}\tau \left(\sum_{n,\mu} + (1-\delta_{n,n+\mu})v(n)v^*(n)T_\mu^*(n) + \mathrm{HC} - \delta_{n,n+\mu}v(n)v^*(n)[\varepsilon_{\mathrm{F}} + \lambda(n;\tau)]\right) = \tilde{A}. \tag{5b}$$

4. Computation of the effective potential

In this section we compute the effective potential (the free energy for the actions given in equations (2*c*) and (2*d*)). The method of calculation will be the same for both actions.

Each time a difference appears we will mention it. In general, for the action in equations (2*d*) and (4*e*–*h*) $D\varphi^+D\varphi$ means De^+De.

Using functional integration techniques [19] we integrate out the fermion degrees of freedom and obtain the effective action Γ as a function of the order parameters $T_\mu(n)$, $B(n, n+\mu)$ and the constraints $\rho_\mu(n)$, $q(n, n+\mu)$, $\lambda(n)$

$$\exp(-\Gamma) = \int D\varphi^+ \, D\varphi \, D\psi^+ \, D\psi \exp(A) \Big/ \int D\varphi^+ \, D\varphi \, D\psi^+ \, D\psi \exp[A(0)] \tag{6a}$$

where $A(0) \equiv A[T_\mu(n) \equiv B(n, n+\mu) \equiv q(n, n+\mu) \equiv \rho_\mu(n) \equiv \lambda(n) \equiv v(n) = 0]$.

The effective action Γ can be separated in terms of amplitudes $\rho^0_\mu(n)$, $r_\mu(n)$, $b(n, n+\mu)$, $q^0(n, n+\mu)$, $\lambda(n)$ and the phase-dependent part $\theta(n, n+\mu)$ and $\chi_\mu(n)$ (for the action in equations (2*d*) and (4*e*–*h*) the order parameter $b(n, n+\mu)$ is absent).

$$\begin{aligned} T_\mu(n) &\equiv r_\mu(n) \exp[i\chi_\mu(n)] \qquad & B(n, n+\mu) &\equiv b(n, n+\mu) \exp[i\theta(n, n+\mu)] \\ \rho_\mu(n) &= \rho^0_\mu(n) \exp[-i\chi_\mu(n)] \qquad & q(n, n+\mu) &= q^0(n, n+\mu) \exp[-i\theta(n, n+\mu)]. \end{aligned} \tag{6b}$$

Performing the functional integration [19] with respect to φ, φ^+ and ψ, ψ^+ we find

$$\begin{aligned} \Gamma = \int_0^\beta d\tau \bigg(& -\frac{1}{2\beta} [\mathrm{tr} \ln(\mathbf{D}^\mathrm{F}_0) + \mathrm{tr} \ln(1 + \mathbf{G}^\mathrm{F}_0 \mathbf{V}^\mathrm{F})] + \frac{1}{\beta} [\mathrm{tr} \ln(\mathbf{D}^\mathrm{B}_0) + \mathrm{tr} \ln(1 + \mathbf{G}^\mathrm{B}_0 \mathbf{V}^\mathrm{B})] \\ & + \sum_{n,\mu} \{\rho_\mu(n;\tau) T^*_\mu(n;\tau) + \rho^*_\mu(n;\tau) T_\mu(n;\tau) \\ & + q(n, n+\mu; \tau) B^*(n, n+\mu; \tau) \\ & + q^*(n, n+\mu; \tau) B(n, n+\mu; \tau) - t[v(n) v^*(n+\mu) T^*_\mu(n) - \mathrm{HC}]\} \\ & + (1 - \delta_{n,n+\mu}) - \delta_{n,n+\mu} \{[\varepsilon_\mathrm{F} + \lambda(n)][1 + |v(n)|^2]\} \bigg). \end{aligned} \tag{7a}$$

$\mathbf{D}^\mathrm{F}_0$ and $\mathbf{D}^\mathrm{B}_0$ are the fermion and boson phase-independent matrices, respectively,

$$\begin{aligned} (\mathbf{G}^\mathrm{F}_0)^{-1} \equiv \mathbf{D}^\mathrm{F}_0 = \{ & \delta_{n,n+\mu}[\hat{\boldsymbol{I}}\partial_\tau - \hat{\boldsymbol{I}}\lambda(n)] + (1 - \delta_{n,n+\mu}) \\ & + \gamma^-_0 \rho^0_\mu(n) + \gamma^+_0 \rho^0_\mu(n) + \gamma^+ q^0(n, n+\mu) + \gamma^- q^0(n, n+\mu)\} \end{aligned} \tag{7b}$$

$$\begin{aligned} (\mathbf{G}^\mathrm{B}_0)^{-1} \equiv \mathbf{D}^\mathrm{B}_0 = [\![& \delta_{n,n+\mu}\{\gamma_0 \partial_\tau + \hat{\boldsymbol{I}}[\tfrac{1}{2}U + \lambda(n)] - \Sigma_3(\varepsilon_\mathrm{F} - \tfrac{1}{2}U)\} \\ & + t(1 - \delta_{n,n+\mu}) \Sigma^-_3 r^0_\mu(n) + \Sigma^+_3 r^0_\mu(n) + \alpha^+ b^0(n, n+\mu) \\ & + \alpha^- b^0(n, n+\mu)]\!]. \end{aligned} \tag{7c}$$

The matrices $\mathbf{V}^\mathrm{F}$ and $\mathbf{V}^\mathrm{B}$ contain the phase-dependent part:

$$\begin{aligned} \mathbf{V}^\mathrm{F} = (1 - \delta_{n,n+\mu}) [\![& \gamma^+_0 \rho^0_\mu(n) \{\exp[-i\chi_\mu(n)] - 1\} + \gamma^-_0 \rho^0_\mu(n) \{\exp[+i\chi_\mu(n)] - 1\} \\ & + \gamma^+ q^0(n, n+\mu) \{\exp[i\theta(n, n+\mu)] - 1\} \\ & + \gamma^- q^0(n, n+\mu) \{\exp[-i\theta(n, n+\mu)] - 1\}]\!] \end{aligned} \tag{7d}$$

$$\begin{aligned} \mathbf{V}^\mathrm{B} = t(1 - \delta_{n,n+\mu}) [\![& \Sigma^+_3 r^0_\mu(n) \{\exp[i\chi_\mu(n)] - 1\} + \Sigma^-_3 r^0_\mu(n) \{\exp[-i\chi_\mu(n)] - 1\} \\ & + \alpha^+ b(n, n+\mu) \{\exp[-i\theta(n, n+\mu)] - 1\} \\ & + \alpha^- b(n, n+\mu) \{\exp[i\theta(n, n+\mu)] - 1\}]\!]. \end{aligned} \tag{7e}$$

When we consider the action given in equations (4*e*)–(4*h*) we replace in equation (7*a*) qB^* with $(1/2J)qq^*$, and equations (7*c*) and (7*e*) become scalar operators since we have only one bose field e, e^*. The effective potential given in equation (7*a*) allows for determination of the amplitude contribution to the effective action and the phase-dependent part. In order to compute the phase-dependent part of the effective potential we expand equation (7*a*) in powers of $\mathbf{V}^{B}$ and $\mathbf{V}^{F}$.

The phase-dependent part is evaluated up to fourth order in $\mathbf{V}^{B}$ and $\mathbf{V}^{F}$. This is performed with the aid of the formula [22]

$$\operatorname{tr}\ln[1+\mathbf{G}_0^{F}\mathbf{V}^{F}]=\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{r}\prod_{i=1}^{r}\sum_{n_i}\int_0^{\beta}\mathrm{d}\tau_i\,\operatorname{tr}[\mathbf{G}_0^{F}(n_1,\tau_1;n_2\tau_2)\mathbf{V}^{F}(n_2,n_3;\tau_2)\ldots\mathbf{G}_0^{F}(n_{r-1},\tau_{r-1},n_r\tau_r)\mathbf{V}_0^{F}(n_r,n_1,\tau_1)]. \tag{7f}$$

Here we use the convention that the trace is over the Pauli matrices. We separate the amplitude and phase part: $\Gamma(r,\rho,b,q,\lambda;\chi,\theta)=\Gamma_0+\Gamma_1$. Γ_0 represents the amplitude part and Γ_1 is the phase part. The phase-dependent part looks similar to models in lattice gauge theory [23].

We will consider first the amplitude part (obtained when we put $\mathbf{V}^{B}=\mathbf{V}^{F}=0$).

Using formula (7*a*) we have for Γ_0:

$$\Gamma_0=\int_0^{\beta}\mathrm{d}\tau\left(-(1/2\beta)\operatorname{tr}\ln[\mathbf{D}_0^{F}]+\frac{1}{\beta}\operatorname{tr}\ln[\mathbf{D}_0^{B}]+L^{d}[-(\varepsilon_F+\lambda)(1+|v|^2)+2b_0q_0+2r_0q_0-t|v_0|^2\eta r_0]\right). \tag{7g}$$

In evaluating Γ_0 we have used: $\rho_\mu(n)=\rho_0$, $q(n,n+\mu)=q_0$, $\lambda(n)=\lambda$, $\mu_\mu(n)=\mu_0$, $b(n,n+\mu)=b_0$, $v(n)=v_0$. The parameter η is obtained after we average with respect to the phase $\chi_\mu(n)$. The fluctuations of the phase are governed by the phase-dependent part functional Γ_1:

$$\eta=2\langle\cos\chi_\mu(n)\rangle_{\Gamma_1}. \tag{7h}$$

When we use the action given in equations (4*e*–*h*), b_0q_0 is replaced by $(1/2J)q_0^2$. In order to find Γ_0 we have to know the spectrum of the matrices $\mathbf{D}_0^{F}$ and $\mathbf{D}_0^{B}$. We consider first the action given in equations (4*a*)–(4*d*).

The eigenvalues of the matrix $\mathbf{D}_0^{F}$ are given by:

$$E(K)=\{[r_0\tau(K)+\lambda]^2+q_0^2\tau(K)^2\}^{1/2} \tag{8a}$$

where

$$\tau(K)=2(\cos K_x+\cos K_y). \tag{8b}$$

A similar diagonalisation of the Bose matrix gives:

$$\operatorname{tr}\ln\mathbf{D}_0^{B}=\frac{1}{\beta}L^{-d}\sum_{K,\omega_n=2\pi n/\beta}\ln\{[\mathrm{i}\omega_n-\Omega^{+}(K)][-\mathrm{i}\omega_n-\Omega^{-}(K)-t^2\tau^2(K)b_0^2]\} \tag{8c}$$

where $\Omega^{+}(K)$ and $\Omega^{-}(K)$ are the hole and double occupancy bands (*e* and *d*, respectively).

$$\Omega^{+}(K)=(\varepsilon_F+\lambda)-t\rho_0\tau(K) \tag{8d}$$

$$\Omega^{-}(K)=U-(\varepsilon_F+\lambda)+t\rho_0\tau(K). \tag{8e}$$

We evaluate equation (7*g*) and find

$$\Gamma_0=-2\sum_K\ln\{\cosh[\beta E(K)/2]+\sum_{K,\omega_n}\ln\{[\mathrm{i}\omega_n-\Omega^{+}(K)][-\mathrm{i}\omega_n-\Omega^{-}(K)-t^2\tau^2(K)b_0^2]\}+\beta L^{d}[-(\varepsilon_F+\lambda)(1+|v_0|^2)-t|v_0|^2\eta r_0+2b_0q_0+2r_0\rho_0]. \tag{9a}$$

Repeating the calculation for the action given in equations (2*d*) (and (4*e*–4*h*), the action obtained after one removes the double occupancy), we obtain instead equation (9*b*):

$$\Gamma_0 = -2\sum_K \ln\{\cosh[\beta E(K)/2]\} + \sum_{K,\omega_n} \ln[\mathrm{i}\omega_n - \Omega^+(K)] + \beta L^d[+(1/J)q_0^2 + 2r_0\rho_0 - (\varepsilon_F + \lambda)(1 + |v_0|^2) - t|v_0|^2\eta r_0]. \qquad (9b)$$

$E(K)$ is given by equation (8a) and $\Omega^+(K)$ by equation (8d).

At the end of this section we give the expression for Γ_1, which is obtained with the aid of formula (7f). Using the matrices $\mathbf{V}^B$ and $\mathbf{V}^F$ we find

$$\begin{aligned}\Gamma_1 = \sum_{n,\mu,\nu} [\![K_1\{1 - \cos[\Delta_\mu\theta(n, n+\nu) - \chi_\mu(n) - \chi_\mu(n+\nu)]\} \\ + K_2\{1 - \cos[\chi_\mu(n) + \chi_\nu(n+\mu) + \chi_{-\mu}(n+\mu+\nu) + \chi_{-\nu}(n+\nu)]\} \\ + K_3\{1 - \cos[\theta(n, n+\mu) - \theta(n+\mu, n+\mu+\nu) \\ + \theta(n+\mu+\nu, n+\nu) - \theta(n+\nu, n)]\}]\!] \end{aligned} \qquad (9c)$$

where $\Delta_\mu\theta(n, n+\nu) = \theta(n, n+\nu) - \theta(n = \mu, n + \mu\sigma + \nu)$. The coefficients K_1, K_2 and K_3 are functions of the amplitude part r, b. These coefficients are determined by perturbation expansion, see equation (7f). The phase fluctuations $\chi_\mu(n)$ are the neutral gauge field induced by the spinons

$$\chi_\mu(n) + \chi_\mu(n+\mu) \approx \int_n^{n+\mu} A(s)\,\mathrm{d}s + \int_{n+\nu}^{n+\nu+\mu} A(s)\,\mathrm{d}s.$$

The functional Γ_1 given in equation (9b) was written in a form used in lattice gauge theory [23].

5. Investigation of the free energy Γ_0

At the mean-field level we now investigate Γ_0, as well as the behaviour of the order parameter $\Sigma\sigma f_\sigma(n) f_{-\sigma}(n+\mu)$ (the variation in Γ_0 with respect to b_0 and q_0 (see equation 9a) and in Γ_0 with respect to q_0 (see equation 9b)).

We start with equation (9a). Performing the variation with respect to $b = b_0$ and $q = q_0$, we find

$$\begin{aligned}\sum_K \frac{\tau^2(K)}{E(K)} \tanh[\tfrac{1}{2}\beta E(K)] &= \left(\sum_K \frac{1}{\beta}\sum_{\omega_n} \frac{2t^2\tau^2(K)}{\{[\mathrm{i}\omega_n - \Omega^+(K)][-\mathrm{i}\omega_n - \Omega^-(K)] - t^2\tau^2(K)b_0^2\}}\right)^{-1} \\ &\simeq \frac{8t^2}{(U + 2\lambda)}\left\{\sum_K \frac{\tau^2(K)}{4}[1 + N_\beta^+(K) + N_\beta^-(K)]\right\}^{-1} \equiv \frac{1}{J_{\text{eff}}}. \end{aligned} \qquad (10a)$$

For $t/U \ll 1$ we find $J_{\text{eff}} \approx 4t^2/U$, a similar result obtained from mapping of the Hubbard model to an antiferromagnet. The function $N_\beta^+(K)$ represents the boson hole occupation function, $N_\beta^+(K) = \{\exp[\beta\Omega^+(K)] - 1\}^{-1}$; $N_\beta^-(K)$ is the double occupation function.

These functions give rise to a thermal normalisation of J_{eff}. The critical temperature is obtained from $q_0(T = T_c) = b_0(T = T_c) = 0$. This justifies the neglect of the term $t^2\tau^2(K)b_0^2$. The neglect will not be correct if we want to find the gap at $T = 0$. Due to this additional term we obtain $J_{\text{eff}}(q_0) \simeq 4t^2/U[1 + \alpha_1(t^2/U)q_0^2 + \ldots](\alpha_1 \simeq 1)$, which shows that there is no simple relation between the gap at $T = 0$ and T_c.

Repeating the calculation with respect to equation (9b) we find the same equation as that in (10a) with the difference that $J_{\text{eff}} = J = 4t^2/U$.

The variation in Γ_0 with respect to λ shows that the constraint condition (equation 1c) is satisfied.

$$\partial\Gamma_0/\partial\lambda = n_F + n_e + n_d - 1 = 0 \tag{10b}$$

where

$$n_F = \frac{1}{L^d}\left\langle \sum_{K,\sigma=\uparrow,\downarrow} f_\sigma^+(K) f_\sigma(K)\right\rangle_{\Gamma_0} \tag{10c}$$

$$n_d = \frac{1}{L^d}\left\langle \sum_K d^+(K)d(K)\right\rangle_{\Gamma_0} \tag{10d}$$

$$n_e = n_e^{K\neq 0} + n_0 \tag{10e}$$

$$n_0 = |v_0|^2 \qquad n_e^{K\neq 0} = \frac{1}{L^d}\left\langle \sum_K e^+(K)e(K)\right\rangle_{\Gamma_0} \tag{10f}$$

(when we use equation (9*b*), n_d is absent in equation (10*b*)). The variation in Γ_0 with respect to ε_F gives the number of particles $N_p = 1 - \delta$.

$$N_p = -(\partial\Gamma_0/\partial\varepsilon_F) = 1 - n_e + n_d. \tag{10g}$$

Since $n_d = 0$ we find from equation (10*b*) that $n_F = 1 - \delta$ (δ is the hole concentration). Using equation (10*c*) we obtain

$$\delta \simeq \frac{1}{L^d}\sum_K \frac{r_0\tau(K) + \lambda}{E(K)} \tanh[\tfrac{1}{2}\beta E(K)]. \tag{10h}$$

Equation (10*h*) shows that $\lambda = \lambda(\delta) \sim \delta$. The solution of equation (10*a*), together with (10*h*), shows that the RVB critical temperature T_c^{RVB} is (in the limit of small hole concentration and $r_0/J_{eff} < 1$):

$$(T_c^{RVB}/J_{eff}) \simeq 1 - (r_0/J_{eff})^2 C_1 \qquad C_1 = 3/4. \tag{11a}$$

A common approximation used in the literature is to substitute $r_0 \simeq \delta$, but this relation is not legitimate in two dimensions because it requires hole condensation, which is absent in two dimensions. In order to test the dependence of T_c on δ we perform a variation in Γ_0 with respect to r_0 and ρ_0. The variation with respect to ρ_0 gives the relation between r_0 and the condensate

$$r_0 = t|v_0|^2\eta = tn_0\eta. \tag{11b}$$

The variation in Γ_0 with respect to r_0 ($\partial\Gamma_0/\partial r_0 = 0$) gives

$$\rho_0 = r_0 \sum_K (\tau^2(K)/E(K)) \tanh[\tfrac{1}{2}\beta E(K)] = r_0 J_{eff}^{-1}. \tag{11c}$$

From equation (10*g*) we have:

$$\delta = n_e = \frac{1}{L^d}\sum_K \{\exp[\beta\Omega^+(K)] - 1\}^{-1}. \tag{11d}$$

Since there is no boson condensation in two dimensions [$\Omega^+(K) = (\varepsilon_F - \lambda - 4t\rho_0) + 2t\rho_0(1 - \cos K_x) + 2t\rho_0(1 - \cos K_y)$], equation (11*d*) diverges for $\varepsilon_F - \lambda - 4t\rho_0 \to 0$). In order to have condensation of hole bosons in the state $K = 0$ we assume the existence of a transfer hopping element in the z-direction. The critical temperature is obtained from equation (4*d*) with the requirement that $\varepsilon_F - \lambda - 4t\rho_0 \to 0$, and we find:

$$T_c^{BE} \simeq C_0 t\rho_0\delta^\alpha = C_0 t(r_0/J_{eff})\delta^\alpha. \tag{12a}$$

The constant C_0 and α depend on the coupling in the third direction. For 3D isotropic case $\alpha = \frac{2}{3}$ and for a two-dimensional system weakly coupled in the third direction, $\alpha \simeq 1$.

From equation (12*a*) we see that T_c^{BE} the condensation temperature depends on r_0 (for a given concentration δ) and from equation (11*a*) we have that T_c^{RVB} also depends on r_0. According to equation (5*a*) the critical superconducting temperature is given by the vanishing of the order parameter.

$$\langle e\rangle^2\langle q_0\rangle = n_0(T_c)\eta(T_c)q_0(T_c) = 0. \tag{12b}$$

At the mean-field level we take η to be a constant. The function $n_0(T)$ represents the hole condensation which vanishes at $T_0 = T_c$.

The critical superconducting temperature is determined after we find the value of r_0 from equations (11*a*) and (12*a*). The critical superconducting temperature is given by:

$$T_c = T_c^{BE} = T_c^{RVB}. \tag{12c}$$

Solving for T_c we find

$$T_c \simeq t(C_0/\sqrt{C_1})\delta^{\alpha}. \tag{12d}$$

This shows that T_c depends on the coupling to the third dimension and increases with δ. We mention that in [25] the authors find also that T_c is proportional to the coupling in the third direction.

6. Fluctuation effects

In this final section we consider the effects of the phase fluctuations given by Γ_1. The analysis relies on the existing results in the literature (see especially the paper by Peskin [11]). From this analysis we will see that for $\delta < \delta_c$ there is no superconducting solution (δ_c is the critical hole concentration).

For $T < T_c$ (T_c is the mean-field value obtained from Γ_0) we compute the parameters; $K_1 \simeq \beta^2 b_0^2(T)r_0^2(T) \simeq \beta^2 J_{\text{eff}}(|T - T_c|/T_c)$; $K_2 \simeq \beta^3 r_0^4(T)$; $K_3 = \beta^3 b_0^4(T)$. For $d = 3$ we find that our results obtained from Γ_0 are stable. The coherence length ξ_s and the penetration length ξ_H are given by [11, 24] $\xi_s^{-1} \simeq \sqrt{K_1(T)}$, $\xi_H^{-1} \simeq \sqrt{K_2(T)}$. From the results of the 3D x–y model coupled to a gauge field [11, 24] (the three-dimensional type II superconductor) we find that in the parameter space (K_1, K_2) we have a type II superconductor for $K_1^{-1} \leqslant K_{1,c}^{-1}$ ($K_{1,c} \simeq 0.33$) and $K_2^{-1} \leqslant K_{2,c}^{-1}$ ($K_{2,c} \simeq 1/13$).

The fact that we have a critical value $K_2 = K_{2,c}$ implies the existence of a critical concentration $\delta = \delta_c$ and we have $T_c \simeq t(\delta - \delta_c)$, $\delta > \delta_c \equiv \delta_c(K_{2,c})$.

For $\delta = 0$, $K_2 = K_1 = 0$ and Γ_1 reduces to a simple lattice gauge theory with the critical behaviour of an x–y model [23]:

$$\Gamma_1|_{\delta=0} \simeq K_3 \sum_{n,\mu,\nu} \{1 - \cos[\theta(n, n + \mu) - \theta(n + \mu, n + \mu + \nu) + \theta(n + \mu + \nu, n + \nu) - \theta(n + \nu, n)]\}.$$

The analysis of the phase-dependent functional shows that we have three correlation lengths. In the insulating phase the RVB correlation length is determined by K_3 ($\delta = 0$, $K_1 = K_2 = 0$, $K_3 \neq 0$, RVB $\simeq [K_3(T)]^{-1/2}$.

In the superconducting phase we have $\xi_s \simeq [K_1(T)]^{-1/2}$ and the spinon correlation length, which we identify with the Higgs correlation length $\xi_H \simeq [K_2(T)]^{-1/2}$ (when $\xi_H \to \infty$ the condensate order parameter $\langle e\rangle = 0$ vanishes $n_0\eta = 0$, $n_0 \neq 0$, $\eta = 0$).

Finally we should like to mention that our results can be understood in terms of the lattice gauge theory. As we have explained, the operator $C_\sigma^+(n)$ is expressed after the elimination of the double occupancy by $c_\sigma^+(n) = f_\sigma^+(n)e(n)$ or, alternatively, $c_\sigma^+(n) = f^+(n)e_\sigma^+(n)$ (a spinless fermion, f, f^+ and a spin-$\frac{1}{2}$ boson e_σ, e_σ^+).

The superconducting order parameter is given by

$$\Delta_\mu(n) \overset{\text{def}}{=} C^+_\sigma(n)C^+_{-\sigma}(n+\mu) = f^+_\sigma(n)[e(n)e(n+\mu)]f^+_{-\sigma}(n+\mu)$$
$$= f^+(n)[e_\sigma(n)e_{-\sigma}(n+\mu)]f^+(n+\mu)$$

with the gauge field $\chi_\mu(n)$ given by $e_\sigma(n)e_{-\sigma}(n+\mu) \simeq \exp(\mathrm{i}\chi_\mu(n))$. Using equation (9*c*) we find in the continuum limit

$$\Gamma_1 \simeq K_1 \sum_{n,\mu} \cos[\Delta_\mu \theta(n) - \chi_\mu(n)] + K_2 \sum_{n,\mu,\nu} \cos[\chi_\mu(n) + \chi_\nu(n+\mu) + \chi_{-\mu}(n+\nu+\mu) + \chi_{-\nu}(n+\nu)]. \quad (13)$$

Assuming that we are in the quantum paramagnet phase (a confined phase) the fluctuations governed by K_3 are negligible.

According to the Abelion–Higgs mechanism, equation 13 has a confining phase in which spontaneous breaking of $U(1)$ symmetry occurs $\langle\exp(\mathrm{i}\theta(n))\rangle = 0$ and as a result superconductivity appears. This is called the Higgs phase with massive $\chi_\mu(n)$ fluctuations.

To conclude, we show that the Hubbard model has a superconducting solution for the non-half-filled band case. This solution exists for 2D system coupled weakly in the third dimension (the third dimension is required even at the mean-field level). Considering phase fluctuations beyond the mean field we have constructed the gauge theory for the Hubbard model and have shown that superconductivity appears in the confining phase with a Higgs field.

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